Term Rewriting — Lecture 3

26S — March 23, 2026

All Solutions

1 The empty multiset over A is defined by $\varnothing(a) = 0$ for all $a \in A$. We have $a \in M$ if and only if $M(a) > 0$. Two multisets $M_1, M_2 \in \mathcal{M}(A)$ are equal if and only if $M_1(a) = M_2(a)$ for all $a \in A$. The intersection $M_1 \cap M_2$ of multisets $M_1, M_2 \in \mathcal{M}(A)$ is defined by $(M_1 \cap M_2)(a) = \min\{M_1(a), M_2(a)\}$.

2 The implication $M_1 >_{\mathsf{mul}} M_2 \implies M_1 \uplus N >_{\mathsf{mul}} M_2 \uplus N$ is an easy consequence of the definition of $>_{\mathsf{mul}}$. So suppose $M_1 \uplus N >_{\mathsf{mul}} M_2 \uplus N$. By definition there exist multisets $X, Y \in \mathcal{M}(A)$ such that $M_2 \uplus N = ((M_1 \uplus N) - X) \uplus Y$, $\varnothing \neq X \subseteq M_1 \uplus N$, and for all $y \in Y$ there exists an $x \in X$ such that $x > y$. According to Exercise A.24 we may assume X and Y to be disjoint. We claim $X \subseteq M_1$. Suppose to the contrary that $X \nsubseteq M_1$. Then there exists an $x \in A$ such that $X(x) > M_1(x)$. Since X and Y are disjoint, we must have $Y(x) = 0$, and hence $(((M_1 \uplus N) - X) \uplus Y)(x) = (M_1(x) + N(x)) - X(x)$. On the other hand we have $(M_2 \uplus N)(x) = M_2(x) + N(x)$. So $M_1(x) - X(x) = M_2(x)$, which is impossible as $X(x) > M_1(x)$ and $M_2(x) \geqslant 0$. Therefore $X \subseteq M_1$ and thus we have

$$((M_1 \uplus N) - X) \uplus Y = (M_1 - X) \uplus Y \uplus N = M_2 \uplus N$$

which implies $(M_1 - X) \uplus Y = M_2$, yielding the desired $M_1 >_{\mathsf{mul}} M_2$.

3 The assignment $\alpha_\odot$ with $\alpha_\odot(x) = \odot$ and $\alpha_\odot(y) = \ominus$ satisfies $[\alpha_\odot]_{\mathcal{B}}((\mathsf{0} + x) + (y + \mathsf{s}(y))) = \odot$ for every $\odot \in \{\oplus, \ominus, \otimes, \oslash\}$.

4 (a) The following proof tree shows $\mathsf{a} \approx_{\mathcal{E}} \mathsf{b}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{f(f(a))} \approx \mathsf{f(a)}}\,[\mathsf{a}] \quad \dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}]}{\mathsf{f(f(a))} \approx \mathsf{a}}\,[\mathsf{t}]}{\mathsf{a} \approx \mathsf{f(f(a))}}\,[\mathsf{s}] \quad \dfrac{}{\mathsf{f(f(a))} \approx \mathsf{g(a,a)}}\,[\mathsf{a}]}{\mathsf{a} \approx \mathsf{g(a,a)}}\,[\mathsf{t}] \quad \dfrac{\dfrac{\dfrac{}{\mathsf{a} \approx \mathsf{a}}\,[\mathsf{r}] \quad \dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}]}{\mathsf{g(a,f(a))} \approx \mathsf{g(a,a)}}\,[\mathsf{c}]}{\mathsf{g(a,a)} \approx \mathsf{g(a,f(a))}}\,[\mathsf{s}]}{\mathsf{a} \approx \mathsf{g(a,f(a))}}\,[\mathsf{t}] \quad \dfrac{}{\mathsf{g(a,f(a))} \approx \mathsf{b}}\,[\mathsf{a}]}{\mathsf{a} \approx \mathsf{b}}\,[\mathsf{t}]$$

According to Theorem 2.4.12 $\mathsf{a} \approx \mathsf{b}$ belongs to the equational theory of $\mathcal{E}$.

(b) Consider the algebra $\mathcal{A}$ with carrier $A = \{\circ, \bullet\}$ and interpretations $\mathsf{a}_{\mathcal{A}} = \mathsf{b}_{\mathcal{A}} = \circ$, $\mathsf{f}_{\mathcal{A}}(\circ) = \circ$, $\mathsf{f}_{\mathcal{A}}(\bullet) = \bullet$, $\mathsf{g}_{\mathcal{A}}(\circ,\circ) = \mathsf{g}_{\mathcal{A}}(\circ,\bullet) = \mathsf{g}_{\mathcal{A}}(\bullet,\bullet) = \circ$, and $\mathsf{g}_{\mathcal{A}}(\bullet,\circ) = \bullet$. We have $\mathsf{f}(x) =_{\mathcal{A}} x$, $\mathsf{f(f(a))} =_{\mathcal{A}} \mathsf{g}(x,x)$, and $\mathsf{g}(x,\mathsf{f}(x)) =_{\mathcal{A}} \mathsf{b}$, so $\mathcal{A}$ is a model for $\mathcal{E}$. The equation $\mathsf{g}(x,y) \approx \mathsf{g}(y,x)$ is not valid in $\mathcal{A}$ because $\mathsf{g}_{\mathcal{A}}(\circ,\bullet) = \circ \neq \bullet = \mathsf{g}_{\mathcal{A}}(\bullet,\circ)$. Hence $\mathsf{g}(x,y) \approx \mathsf{g}(y,x)$ does not belong to the equational theory of $\mathcal{E}$.

(c) The following proof tree shows $\mathsf{g(f(a),a)} \approx_{\mathcal{E}} \mathsf{f(b)}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}] \quad \dfrac{\dfrac{}{\mathsf{f(a)} \approx \mathsf{a}}\,[\mathsf{a}]}{\mathsf{a} \approx \mathsf{f(a)}}\,[\mathsf{s}]}{\mathsf{g(f(a),a)} \approx \mathsf{g(a,f(a))}}\,[\mathsf{c}] \quad \dfrac{}{\mathsf{g(a,f(a))} \approx \mathsf{b}}\,[\mathsf{a}]}{\mathsf{g(f(a),a)} \approx \mathsf{b}}\,[\mathsf{t}] \quad \dfrac{\dfrac{}{\mathsf{f(b)} \approx \mathsf{b}}\,[\mathsf{a}]}{\mathsf{b} \approx \mathsf{f(b)}}\,[\mathsf{s}]}{\mathsf{g(f(a),a)} \approx \mathsf{f(b)}}\,[\mathsf{t}]$$

According to Theorem 2.4.12 the equation $\mathsf{g}(\mathsf{f}(\mathsf{a}),\mathsf{a}) \approx \mathsf{f}(\mathsf{b})$ belongs to the equational theory of $\mathcal{E}$.

5 First we show that the symmetry and transitivity inference rules of equational logic are consequences of the inference rule

$$\frac{s \approx t \qquad u \approx t}{s \approx u}$$

Symmetry is proved using reflexivity:

$$\frac{[\mathsf{r}]\,\dfrac{}{t \approx t} \qquad s \approx t}{t \approx s}$$

Transitivity is proved as follows:

$$\frac{s \approx t \qquad \dfrac{[\mathsf{r}]\,\dfrac{}{u \approx u} \qquad t \approx u}{u \approx t}}{s \approx u}$$

Conversely, the new inference rule is a consequence of symmetry and transitivity:

$$[\mathsf{t}]\,\frac{s \approx t \qquad \dfrac{u \approx t}{t \approx u}\,[\mathsf{s}]}{s \approx u}$$