universität innsbruck

All Solutions

1 (a) First we create the following sets:

$$\{\mathsf{a}\} \qquad \{\mathsf{b}\} \qquad \{\mathsf{f}(\mathsf{a})\} \qquad \{\mathsf{f}(\mathsf{b})\} \qquad \{\mathsf{c}\}$$

Next we merge sets using the equations of $\mathcal{E}$:

$$\{\mathsf{a},\mathsf{b},\mathsf{f}(\mathsf{a})\} \qquad \{\mathsf{f}(\mathsf{b}),\mathsf{c}\}$$

Since a and b belong to the same set but $\mathsf{f}(\mathsf{a})$ and $\mathsf{f}(\mathsf{b})$ do not, we merge the two sets:

$$\{\mathsf{a},\mathsf{b},\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}),\mathsf{c}\}$$

Since a and c belong to the same set, the equation $\mathsf{a} \approx \mathsf{c}$ is valid in $\mathcal{E}$.

(b) Consider the algebra $\mathcal{A}$ with carrier $A = \{\bot, \top\}$ and interpretations $\mathsf{a}_\mathcal{A} = \mathsf{b}_\mathcal{A} = \mathsf{c}_\mathcal{A} = \mathsf{f}_\mathcal{A}(x) = \bot$ for all $x \in A$. The algebra $\mathcal{A}$ is a model of $\mathcal{E}$ as the interpretation of each left- and right-hand side of every equation is $\bot$. The equation $\mathsf{f}(x) \approx x$ is not valid on $\mathcal{A}$ because $[\alpha]_\mathcal{A}(\mathsf{f}(x)) = \mathsf{f}_\mathcal{A}(\top) = \bot \neq \top = \alpha(x) = [\alpha]_\mathcal{A}(x)$ for the assignment α defined by $\alpha(x) = \top$.

2 (a) The rewrite rules give rise to the following constraints:

$$\begin{aligned} y + 1 &> y & y + 1 &> 0 & 2x + y + 3 &> 2x + y + 2 \\ 2x + y + 3 &> 2x + y + 2 & x + 2 &> x + 1 & 4x + 2y + 6 &> 4x + 2y + 5 \\ & & x + y + 3 &> x + y + 1 & & \end{aligned}$$

Since the inequalities hold for all $x, y \in \mathbb{N}$, the interpretation is compatible with $\mathcal{R}$.

(b) We obtain the following constraints:

$$\begin{aligned} y + 1 &> y & 2y + a &> 0 \\ 2x + y + 3 &> 2x + y + 2 & x + a + 1 &> x + 1 \\ & & x + 2y + a + 3 &> x + 2y + a \\ 3x + by + c &> 3x + 4y + 3a & & \\ dx + ey + f &> 8x + (2b+1)y + 2c + a + 1 & & \end{aligned}$$

We have to find natural numbers a, b, c, d, e, and f such that these inequalities hold for all $x, y \in \mathbb{N}$. One readily checks that we can take $a = 1$, $b = c = 4$, $d = 8$, $e = 9$, and $f = 11$.

(c) [1] The rule $\mathsf{min}(x, y) \to x \dot{-} (x \dot{-} y)$ gives rise to the constraint $5x + y > 4x + y$, which is violated for $x = y = 0$. Hence the interpretation cannot be used to prove the termination of $\mathcal{R}$.

$\boxed{2}$ The rewrite rules give rise to the following constraints:

$$\begin{aligned} y+3 &> y \\ 2x+y+3 &> 2x+y+1 \end{aligned} \qquad \begin{aligned} y+2 &> 1 \\ 2x+3 &> x+1 \\ 2x+y+3 &> 2x+y \end{aligned}$$

$$\begin{aligned} 4x^2+y+4 &> 4x+y \\ 5x^2+5x+5y+5 &> 4x^2+6x+3y+4 \end{aligned}$$

Since the inequalities hold for all $x, y \in \mathbb{N}$, the interpretation proves the termination of $\mathcal{R}$. (Note that $5x^2+5x \geqslant 4x^2+6x$ for all $x \in \mathbb{N}$ and $5y+5 > 3y+4$ for all $y \in \mathbb{N}$.)

$\boxed{3}$ The rule $\mathsf{s}(x)+y \to \mathsf{s}(x+y)$ gives rise to the constraint $6x+y+3 > 6x+2y+1$, which is violated for $x=0$ and $y=2$. Hence the interpretation cannot be used to prove the termination of $\mathcal{R}$.

3 Consider the well-founded monotone algebra $(\mathcal{A}, >_{\mathbb{N}})$ with carrier $\mathbb{N} \setminus \{0\}$ and interpretations $\mathsf{f}_{\mathcal{A}}(x,y) = 3^x + y$ and $\mathsf{g}_{\mathcal{A}}(x) = x+1$. Since

$$\begin{aligned} 3^{x+1}+y &>_{\mathbb{N}} 3^x + (3^x + y) + 1 \\ 3^x + x &>_{\mathbb{N}} x+2 \end{aligned}$$

for all $x, y \in \mathbb{N} \setminus \{0\}$, the TRS is terminating.

4 Yes. By taking the polynomial interpretation $\mathsf{0}_{\mathbb{N}} = 0$, $\mathsf{s}_{\mathbb{N}}(x) = x+1$, and $\mathsf{f}_{\mathbb{N}}(x) = 2x^2 - x + 1$, the rewrite rules are transformed into the following constraints:

$$\begin{aligned} 1 &> 0 \\ 2 &> 1 \\ 7 &> 6 \end{aligned} \qquad \begin{aligned} 2 &> 1 \\ 3 &> 2 \\ 8 &> 7 \end{aligned}$$

Note that $\mathsf{f}_{\mathbb{N}}$ is well-defined (i.e., $\mathsf{f}_{\mathbb{N}}(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$) and strictly monotone.