universität innsbruck

Term Rewriting — Lecture 8

26S — May 11, 2026

All Solutions

1 Use the weight function $w(\mathsf{average}) = 0$ and $w(\mathsf{0}) = w(\mathsf{s}) = w_0 = 1$ together with the empty precedence. All rules are weight-decreasing, except $\mathsf{average}(\mathsf{s}(x), y) \to \mathsf{average}(x, \mathsf{s}(y))$. The latter rule is weight-preserving. Since $\mathsf{s}(x) >_{\mathsf{kbo}} x$ by clause (a) of Definition 4.4.7, $\mathsf{average}(\mathsf{s}(x), y) >_{\mathsf{kbo}} \mathsf{average}(x, \mathsf{s}(y))$ ⟨2, 1⟩.

2 Use the weight function $w(\mathsf{p1}) = w(\mathsf{p2}) = w(\mathsf{p5}) = w(\mathsf{p10}) = w(+) = w_0 = 1$ together with the precedence $\mathsf{p10} > \mathsf{p5} > \mathsf{p2} > \mathsf{p1}$. The rewrite rules

$$\mathsf{p1} + \mathsf{p1} \to \mathsf{p2} \qquad \mathsf{p1} + (\mathsf{p1} + x) \to \mathsf{p2} + x$$
$$\mathsf{p5} + \mathsf{p5} \to \mathsf{p10} \qquad \mathsf{p5} + (\mathsf{p5} + x) \to \mathsf{p10} + x$$
$$\mathsf{p1} + (\mathsf{p2} + \mathsf{p2}) \to \mathsf{p5} \qquad \mathsf{p1} + (\mathsf{p2} + (\mathsf{p2} + x)) \to \mathsf{p5} + x$$
$$\mathsf{p2} + (\mathsf{p2} + \mathsf{p2}) \to \mathsf{p1} + \mathsf{p5} \qquad \mathsf{p2} + (\mathsf{p2} + (\mathsf{p2} + x)) \to \mathsf{p1} + (\mathsf{p5} + x)$$

are weight-decreasing. All other rewrite rules are weight-preserving and satisfy $\ell >_{\mathsf{kbo}} r$ ⟨2, 1⟩.

3 (a) The TRSs $\mathcal{R}_2$ and $\mathcal{R}_5$ are canonical. The other TRSs are not reduced and thus also not canonical.

(b) We have

conversion equivalent

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$
$\mathcal{R}_2$	$\times_1$				
$\mathcal{R}_3$	$\checkmark_4$	$\times_1$			
$\mathcal{R}_4$	$\times_1$	$\checkmark_5$	$\times_1$		
$\mathcal{R}_5$	$\checkmark_2$	$\times_1$	$\checkmark_3$	$\times_1$	
$\mathcal{R}_6$	$\times_1$	$\checkmark_6$	$\times_1$	$\checkmark_7$	$\times_1$

normalization equivalent

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$
$\mathcal{R}_2$	$\times_1$				
$\mathcal{R}_3$	$\times_8$	$\times_1$			
$\mathcal{R}_4$	$\times_1$	$\times_9$	$\times_1$		
$\mathcal{R}_5$	$\times_8$	$\times_1$	$\checkmark_3$	$\times_1$	
$\mathcal{R}_6$	$\times_1$	$\times_9$	$\times_1$	$\times_9$	$\times_1$

with the following explanations:

[1] For $\mathcal{R}_2$, $\mathcal{R}_4$, and $\mathcal{R}_6$ we have $\mathsf{g}(\mathsf{g}(x)) \to^! \mathsf{g}(x)$ and thus also $\mathsf{g}(\mathsf{g}(x)) \leftrightarrow^* \mathsf{g}(x)$, but this does not hold for $\mathcal{R}_1$, $\mathcal{R}_3$, and $\mathcal{R}_5$.

[2] The TRSs $\mathcal{R}_1$ and $\mathcal{R}_1' = \mathcal{R}_1 \setminus \{\mathsf{a} \to \mathsf{g}(\mathsf{b})\}$ are obviously conversion equivalent. Using LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{a} > \mathsf{b}$, $\mathcal{R}_1'$ can be completed into $\mathcal{R}_5$. Hence $\mathcal{R}_1$ and $\mathcal{R}_5$ are conversion equivalent.

[3] The TRS $\mathcal{R}_3$ is complete and $\mathcal{R}_5 = \ddot{\mathcal{R}}_3$. Hence $\mathcal{R}_3$ and $\mathcal{R}_5$ are (normalization) equivalent according to Theorem 5.3.4.

[4] This follows from items [2] and [3], since conversion equivalence is a transitive relation.

[5] The TRSs $\mathcal{R}_4$ and $\mathcal{R}_4' = (\mathcal{R}_4 \setminus \{\mathsf{a} \to \mathsf{b}\}) \cup \{\mathsf{b} \to \mathsf{a}\}$ are obviously conversion equivalent. Using LPO with precedence $\mathsf{f} > \mathsf{b} > \mathsf{a}$, $\mathcal{R}_4'$ can be completed into $\mathcal{R}_2$. Hence $\mathcal{R}_2$ and $\mathcal{R}_4$ are conversion equivalent.

[6] The TRSs $\mathcal{R}_6$ and $\mathcal{R}_6' = (\mathcal{R}_6 \setminus \{\mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{g}(\mathsf{b})\}) \cup \{\mathsf{b} \to \mathsf{a}\}$ are conversion equivalent since $\mathsf{b} \leftarrow \cdot \to \mathsf{g}(\mathsf{b})$ holds in $\mathcal{R}_6'$. Using LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$, $\mathcal{R}_6'$ is completed into the canonical TRS $\mathcal{R}_2$. Hence $\mathcal{R}_2$ and $\mathcal{R}_6$ are conversion equivalent.

[7] This follows from items [5] and [6], since conversion equivalence is a transitive relation.

8. The term a has no normal form in $\mathcal{R}_1$. Since $\mathcal{R}_3$ and $\mathcal{R}_5$ are terminating, they cannot be normalization equivalent to $\mathcal{R}_1$.

9. We have $\mathsf{a} \to^! \mathsf{a}$ in $\mathcal{R}_2$ but not in $\mathcal{R}_4$ and $\mathcal{R}_6$. Furthermore, $\mathsf{b} \to^! \mathsf{b}$ holds in $\mathcal{R}_4$ but not in $\mathcal{R}_6$.

4 (a) Consider the KB run

		equation	removed at	rule	removed at
1				$\mathsf{TCAT} \xrightarrow{1} \mathsf{T}$	23
2				$\mathsf{GAG} \xrightarrow{2} \mathsf{AG}$	15
3				$\mathsf{CTC} \xrightarrow{3} \mathsf{TC}$	10
4				$\mathsf{AGTA} \xrightarrow{4} \mathsf{A}$	19
5				$\mathsf{TAT} \xrightarrow{5} \mathsf{CT}$	9
6	deduce(1,1,3)	$\mathsf{CT} \approx \mathsf{TCAT}$	7		
7	simplify(6,1)	$\mathsf{CT} \approx \mathsf{T}$	8		
8	orient(7)			$\mathsf{CT} \xrightarrow{8} \mathsf{T}$	
9	compose(5,8)			$\mathsf{TAT} \xrightarrow{9} \mathsf{T}$	
10	collapse(3,8)	$\mathsf{TC} \approx \mathsf{TC}$	11		
11	delete(10)				
12	deduce(4,1,2)	$\mathsf{GA} \approx \mathsf{AGTA}$	13		
13	simplify(12,4)	$\mathsf{GA} \approx \mathsf{A}$	14		
14	orient(13)			$\mathsf{GA} \xrightarrow{14} \mathsf{A}$	
15	collapse(2,14)	$\mathsf{AG} \approx \mathsf{AG}$	16		
16	delete(15)				
17	deduce(9,11,4)	$\mathsf{AGT} \approx \mathsf{AT}$	18		
18	orient(17)			$\mathsf{AGT} \xrightarrow{18} \mathsf{AT}$	
19	collapse(4,18)	$\mathsf{ATA} \approx \mathsf{A}$	19		
20	orient(19)			$\mathsf{ATA} \xrightarrow{20} \mathsf{A}$	
21	deduce(20,11,1)	$\mathsf{TCA} \approx \mathsf{TA}$	21		
22	orient(21)			$\mathsf{TCA} \xrightarrow{22} \mathsf{TA}$	
23	collapse(1,22)	$\mathsf{TAT} \approx \mathsf{T}$	24		
24	simplify(23,9)	$\mathsf{T} \approx \mathsf{T}$	25		
25	delete(24)				

which uses LPO with precedence $\mathsf{A} > \mathsf{C}$ as reduction order. The six remaining rewrite rules constitute an SRS $\mathcal{R}'$

$$\mathsf{CT} \to \mathsf{T} \qquad \mathsf{TAT} \to \mathsf{T} \qquad \mathsf{GA} \to \mathsf{A} \qquad \mathsf{AGT} \to \mathsf{AT} \qquad \mathsf{ATA} \to \mathsf{A} \qquad \mathsf{TCA} \to \mathsf{TA}$$

whose (prime) critical pairs are easily shown to be joinable and hence the above run is fair. According to Theorem 5.4.10 $\mathcal{R}'$ is a complete SRS that is (conversion) equivalent to $\mathcal{R}$. Since $\mathcal{R}'$ is reduced, it is also canonical.

(b) We have

$$\mathsf{TAGCTAGCTAGCT} \xrightarrow[\mathcal{R}']{!} \mathsf{T} \xleftarrow[\mathcal{R}']{!} \mathsf{CTGACTGACT}$$

for the SRS $\mathcal{R}'$ of part (a). Since $\mathcal{R}$ and $\mathcal{R}'$ are (conversion) equivalent,

$$\mathcal{R} \vDash \mathsf{TAGCTAGCTAGCT} \approx \mathsf{CTGACTGACT}$$

holds.

(c) We have

$$\mathsf{TAGCTAGCTAGCT} \xrightarrow[\mathcal{R}']{!} \mathsf{T} \neq \mathsf{TGT} \xleftarrow[\mathcal{R}']{!} \mathsf{CTGCTACTGACT}$$

for the SRS $\mathcal{R}'$ of part (a). Since $\mathcal{R}$ and $\mathcal{R}'$ are (conversion) equivalent,

$$\mathcal{R} \vDash \mathsf{TAGCTAGCTAGCT} \approx \mathsf{CTGCTACTGACT}$$

does not hold.

5 (a) Let $>$ be a well-founded precedence and (w, w_0) be an admissible weight function. We define a new weight function $(w^1, 1)$ with $w^1(f) = w(f) + (n-1)\cdot(w_0 - 1)$ for every n-ary function symbol f. Obviously $w^1(f) \geqslant 0$ for all $f \in \mathcal{F}$ and in particular $w^1(c) \geqslant 1$ for constants $c \in \mathcal{F}$ since $w(c) \geqslant w_0$. Note that $w^1(f) = w(f)$ for every unary function symbol $f \in \mathcal{F}$. Hence $(w^1, 1)$ is admissible for $>$. We show $w(t) = w^1(t) + w_0 - 1$ by induction on t. If $t \in \mathcal{V}$ then $w(t) = w_0 = w^1(t) + w_0 - 1$ since $w^1(t) = 1$. Suppose $t = f(t_1, \dots, t_n)$. We have

$$\begin{aligned} w(t) &= w(f) + w(t_1) + \cdots + w(t_n) = w(f) + w^1(t_1) + w_0 - 1 + \cdots + w^1(t_n) + w_0 - 1 \\ &= w^1(f) - (n-1)\cdots(w_0 - 1) + w^1(t_1) + w_0 - 1 + \cdots + w^1(t_n) + w_0 - 1 \\ &= w^1(f) + w^1(t_1) + \cdots + w^1(t_n) + w_0 - 1 = w^1(t) + w_0 - 1 \end{aligned}$$

where the induction hypothesis is used in the second step. It follows that $w(s) > w(t)$ if and only if $w^1(s) > w^1(t)$. Hence the order $>_{\mathsf{kbo}}$ is not affected when changing (w, w_0) to $(w^1, 1)$.

(b) Let $>$ be a well-founded precedence and (w, w_0) be an admissible weight function. Let $k > 0$. We define a new weight function (w^k, k) with $w^k(f) = w^1(f) \cdot k$ for every function symbol f. Here $(w^1, 1)$ is the weight function defined in the solution of part (a). We have $w(f) \neq 0$ if and only if $w_k(f) = w^1(f) \cdot k \neq 0$, for every unary function symbol f. Hence (w^k, k) is admissible for $>$. A straightforward induction proof reveals $w^k(t) = w^1(t) \cdot k$ for all terms. Hence $w(s) > w(t)$ if and only if $w^k(s) > w^k(t)$. Hence the order $>_{\mathsf{kbo}}$ is not affected when changing (w, w_0) to (w^k, k).

(c) Well-foundedness of $>_{\mathsf{kbo}}$ is lost if we allow $w_0 = 0$. Consider the TRS $\mathcal{R}$ consisting of the single rewrite rule

$$\mathsf{f}(\mathsf{f}(\mathsf{a}, \mathsf{a}), \mathsf{a}) \to \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{a}, \mathsf{a}), \mathsf{a}))$$

and the (illegal) weight function $(w, 0)$ with $w(\mathsf{a}) = w(\mathsf{f}) = 0$ together with the precedence $\mathsf{f} > \mathsf{a}$. The admissibility condition is trivially satisfied. We have $\mathsf{f}(\mathsf{f}(\mathsf{a}, \mathsf{a}), \mathsf{a}) >_{\mathsf{kbo}} \mathsf{f}(\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{a}, \mathsf{a}), \mathsf{a}))$ $\langle 2, 1\rangle$ but $\mathcal{R}$ is not terminating.