

All Solutions

1	\twoheadrightarrow	\rightarrow	\rightarrow^*
1	✓	✓	×
2	✓	✓	×
3	×	✓	✓
4	×	×	×

- 2 (a) There are eight different multi-steps starting from $S(KS)(II)(IK)(KK)$ (the contracted redexes are indicated by underlining):
- $\underline{S}(KS)(II)(IK)(KK) \twoheadrightarrow KS(IK)(II(IK))(KK)$
 - $\underline{S}(KS)(II)(IK)(KK) \twoheadrightarrow KS(IK)(I(IK))(KK)$
 - $\underline{S}(KS)(II)(IK)(KK) \twoheadrightarrow KSK(IIK)(KK)$
 - $\underline{S}(KS)(II)(IK)(KK) \twoheadrightarrow KSK(IK)(KK)$
 - $S(KS)(II)(IK)(KK) \twoheadrightarrow S(KS)I(IK)(KK)$
 - $S(KS)(II)(IK)(KK) \twoheadrightarrow S(KS)(II)K(KK)$
 - $S(KS)(II)(IK)(KK) \twoheadrightarrow S(KS)IK(KK)$
 - $S(KS)(II)(IK)(KK) \twoheadrightarrow S(KS)(II)(IK)(KK)$
- (b) The multi-step $S(KS)(II)(IK)(KK) \twoheadrightarrow KSK(IK)(KK)$ is the only maximal one.

- 3 (a) The SRS admits two critical pairs:

$$abaa \approx aaba$$

$$caaa \approx cababa$$

We have

$$abaa \rightarrow aaa \leftarrow aaba$$

$$caaa \rightarrow cabaa \leftarrow cababa$$

and hence the SRS is strongly closed. Since every SRS is linear, confluence follows from Theorem 6.3.2.

- (b) The given TRS \mathcal{R} is linear. There are four (modulo symmetry) critical pairs:

$$0 \approx 0 \quad x \approx \max(0, x) \quad y \approx \max(y, 0) \quad s(\max(x, y)) \approx \max(s(y), s(x))$$

Since

$$x \leftarrow \max(0, x) \quad y \leftarrow \max(y, 0) \quad s(\max(x, y)) \rightarrow s(\max(y, x)) \leftarrow \max(s(y), s(x))$$

\mathcal{R} is strongly closed and thus confluent by Theorem 6.3.2.

- 4 (a) Yes. The rewrite rules $x \wedge (y \vee z) \rightarrow (x \wedge y) \vee (x \wedge z)$ and $(x' \vee y') \wedge z' \rightarrow (x' \wedge z') \vee (y' \wedge z')$ overlap at the root, resulting in the overlays $((x' \vee y') \wedge y) \vee ((x' \vee y') \wedge z) \leftarrow \twoheadrightarrow (x' \wedge (y \vee z)) \vee (y' \wedge (y \vee z))$ and $(x' \wedge (y \vee z)) \vee (y' \wedge (y \vee z)) \leftarrow \twoheadrightarrow ((x' \vee y') \wedge y) \vee ((x' \vee y') \wedge z)$.

- (b) Let $s \leftarrow \bowtie \rightarrow t$ be an overlay. So there exists a root overlap $\langle \ell_1 \rightarrow r_1, \epsilon, \ell_2 \rightarrow r_2 \rangle$ and an mgu σ of ℓ_1 and ℓ_2 such that $s = \ell_2 \sigma [r_1 \sigma]_\epsilon = r_1 \sigma$ and $t = r_2 \sigma$. It follows that $t \leftarrow \bowtie \rightarrow s$ is on overlay origination from the root overlap $\langle \ell_2 \rightarrow r_2, \epsilon, \ell_1 \rightarrow r_1 \rangle$ and the mgu σ .
- (c) Let $\langle \ell_1 \rightarrow r_1, p, \ell_2 \rightarrow r_2 \rangle$ be an overlap in a CS. Since $p \in \mathcal{Pos}_{\mathcal{F}}(\ell_1)$, $\ell_1(p)$ is a defined function symbol. In a CS this is only possible when $p = \epsilon$. Hence the overlap is a root overlap. It follows that every critical pair of a CS is an overlay.
- 5 (a) The constant \mathbf{a} admits the infinite rewrite sequence $\mathbf{a} \rightarrow \mathbf{g}(\mathbf{a}) \rightarrow \mathbf{g}(\mathbf{g}(\mathbf{a})) \rightarrow \dots$ in \mathcal{R}_1 . Since this is the only (maximal) rewrite sequence starting at \mathbf{a} , \mathcal{R}_1 is not normalizing. The same reasoning applies to the term $\mathbf{g}(\mathbf{a})$ in \mathcal{R}_2 : $\mathbf{g}(\mathbf{a}) \rightarrow \mathbf{f}(\mathbf{a}, \mathbf{g}(\mathbf{a})) \rightarrow \mathbf{f}(\mathbf{a}, \mathbf{f}(\mathbf{a}, \mathbf{g}(\mathbf{a}))) \rightarrow \dots$.
- (b) Consider the TRS \mathcal{R} consisting of the rewrite rules

$$\mathbf{f}(x, x) \rightarrow \mathbf{g}(x) \qquad \mathbf{f}(x, \mathbf{g}(x)) \rightarrow \mathbf{a} \qquad \mathbf{h}(\mathbf{b}, y) \rightarrow \mathbf{f}(\mathbf{h}(y, \mathbf{b}), \mathbf{h}(y, y))$$

The TRS lacks critical pairs but is not confluent:

$$\begin{aligned} \mathbf{h}(\mathbf{b}, \mathbf{b}) &\rightarrow \mathbf{f}(\mathbf{h}(\mathbf{b}, \mathbf{b}), \mathbf{h}(\mathbf{b}, \mathbf{b})) \rightarrow \mathbf{f}(\mathbf{h}(\mathbf{b}, \mathbf{b}), \mathbf{f}(\mathbf{h}(\mathbf{b}, \mathbf{b}), \mathbf{h}(\mathbf{b}, \mathbf{b}))) \rightarrow \mathbf{f}(\mathbf{h}(\mathbf{b}, \mathbf{b}), \mathbf{g}(\mathbf{h}(\mathbf{b}, \mathbf{b}))) \rightarrow \mathbf{a} & (*) \\ \mathbf{h}(\mathbf{b}, \mathbf{b}) &\rightarrow \mathbf{f}(\mathbf{h}(\mathbf{b}, \mathbf{b}), \mathbf{h}(\mathbf{b}, \mathbf{b})) \rightarrow \mathbf{g}(\mathbf{h}(\mathbf{b}, \mathbf{b})) \rightarrow^* \mathbf{g}(\mathbf{a}) \end{aligned}$$

where the sequence $\mathbf{g}(\mathbf{h}(\mathbf{b}, \mathbf{b})) \rightarrow^* \mathbf{g}(\mathbf{a})$ follows from (*). By induction on t we show that every term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ has a normal form.

- ▷ If $t = \mathbf{a}$, $t = \mathbf{b}$, or $t \in \mathcal{V}$ then t is a normal form.
- ▷ If $t = \mathbf{g}(u)$ then $u \rightarrow^! v$ by the induction hypothesis. Hence $t \rightarrow^* \mathbf{g}(v)$ with normal form $\mathbf{g}(v)$.
- ▷ If $t = \mathbf{f}(u, u')$ then $u \rightarrow^! v$ and $u' \rightarrow^! v'$ by the induction hypothesis. Hence $t \rightarrow^* \mathbf{f}(v, v')$. We distinguish three cases. If $v = v'$ then $\mathbf{f}(v, v') \rightarrow \mathbf{g}(v)$ and $\mathbf{g}(v)$ is a normal form. If $v' = \mathbf{g}(v)$ then $\mathbf{f}(v, v') \rightarrow \mathbf{b}$ with normal form \mathbf{b} . In the remaining case $\mathbf{f}(v, v')$ is a normal form.
- ▷ If $t = \mathbf{h}(u, u')$ then $u \rightarrow^! v$ and $u' \rightarrow^! v'$ by the induction hypothesis. Hence $t \rightarrow^* \mathbf{h}(v, v')$. Again we distinguish three cases. If $v \neq \mathbf{b}$ then $\mathbf{h}(v, v')$ is a normal form. If $v = \mathbf{b}$ and $v' \neq \mathbf{b}$ then $\mathbf{h}(v, v') \rightarrow \mathbf{f}(\mathbf{h}(v', \mathbf{b}), \mathbf{h}(v', v'))$ and $\mathbf{f}(\mathbf{h}(v', \mathbf{b}), \mathbf{h}(v', v'))$ is a normal form. In the remaining case we have $v = v' = \mathbf{b}$ and thus $\mathbf{h}(v, v') \rightarrow^! \mathbf{a}$ by (*).