Reachability for Termination*

4th Austria-Japan Summer Workshop on Term Rewriting

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* Supported by Austrian Science Fund (FWF) Y-757

Reachability for Termination of Term Rewriting

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Reachability in the Dependency Framework for Termination of Term Rewriting

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Terminology proposal

$s \to_{\mathcal{R}}^{*} t$ — t is ~~reachable~~ **reached** from s

$\exists\theta.\ s\theta \to_{\mathcal{R}}^{*} t\theta$ — t is **reachable** from s [Sternagel & Sternagel '16]

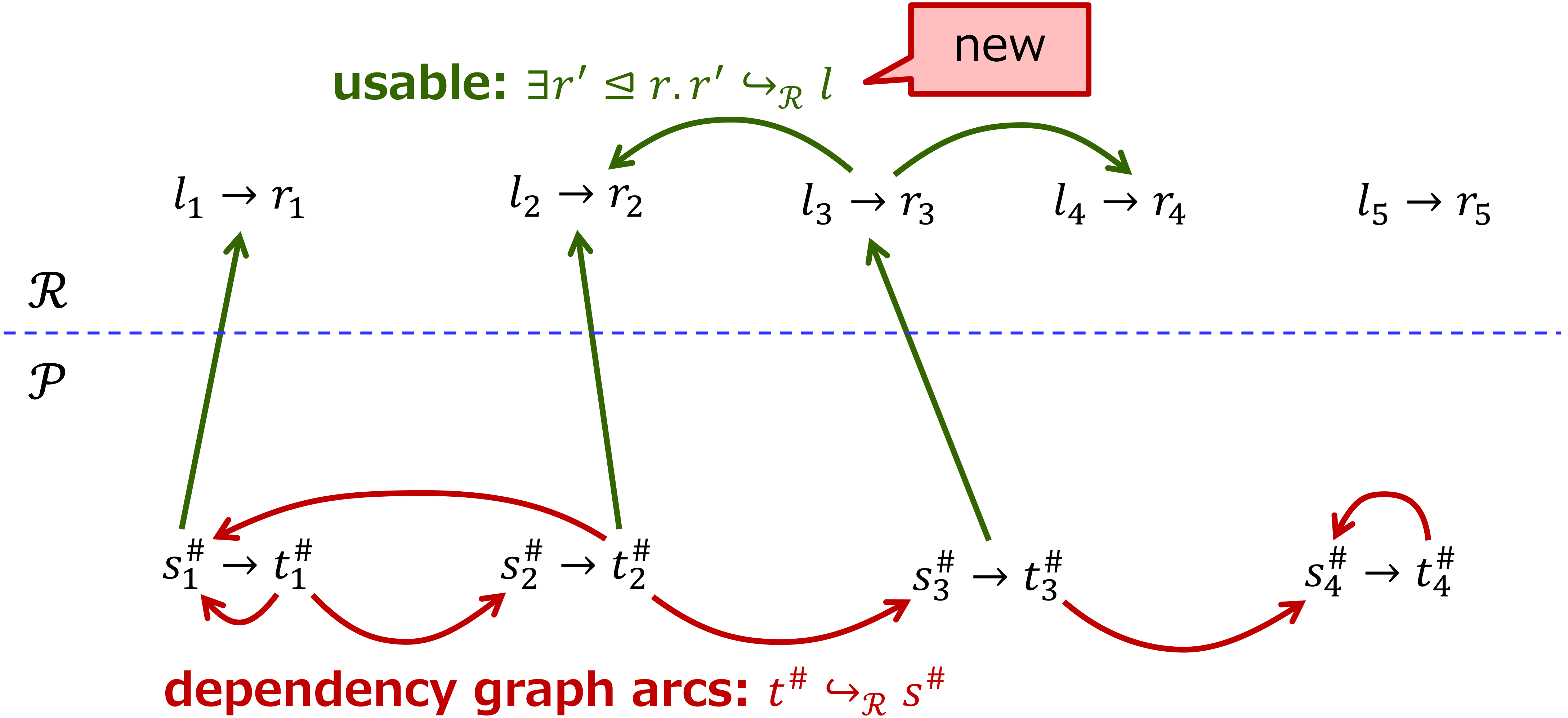
$(s \hookrightarrow_{\mathcal{R}} t)$

Example:

Q: $\mathrm{start}(\mathit{some\_input}) \hookrightarrow_{\mathcal{R}} \mathrm{error}(\mathit{some\_code})$?

A: yes, $\mathrm{start}(5) \to_{\mathcal{R}}^{*} \mathrm{error}(\mathrm{DIV0})$

Reachability in DP framework



Usable rules (before)

Theorem ([Hirokawa & Middeldorp '04 / Giesl+ '05]):

$$\phi(s) := \bigwedge_{f(s_1,\dots) \trianglelefteq_\pi s} \; \bigwedge_{f(l_1,\dots)\to r \in \mathcal{R}} f(l_1,\dots) \to r \in \mathcal{U}$$

If $\phi(\mathrm{rhds}\,\mathcal{P}) \wedge \phi(\mathrm{rhds}\,\mathcal{U})$, then one can ignore $\mathcal{R} \setminus \mathcal{U}$

Theorem ([Sternagel & Thiemann '10]): same for

$$\phi(s) := \bigwedge_{f(s_1,\dots) \trianglelefteq_\pi s} \; \bigwedge_{f(l_1,\dots)\to r \in \mathcal{R}} \mathrm{tcap}(s_1) \sim_{\mathrm{unif}} l_1, \dots \Rightarrow f(l_1,\dots) \to r \in \mathcal{U}$$

Usable rules via reachability

Theorem (new): Usable rules technique applies for

$$\phi(s) := \bigwedge_{f(s_1,\dots)\trianglelefteq_\pi s} \; \bigwedge_{f(l_1,\dots)\to r\in\mathcal{R}} s_1 \hookrightarrow_{\mathcal{R}} l_1, \dots \Rightarrow f(l_1, \dots) \to r \in \mathcal{U}$$

Proof:

I changed original proofs until Isabelle somehow accepted.
So it must be true.

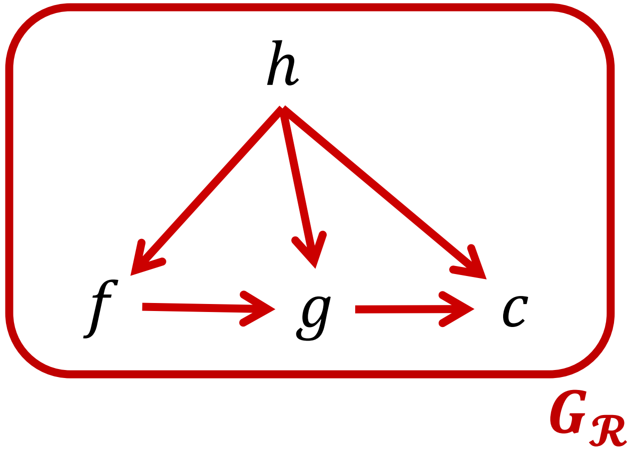
TODO: Understand why the proof works

Estimating reachability

- Requirements
  - efficiency: can't be as hard as termination proving
  - completeness: if t is reachable from s, then it must say so
  - soundness (only for nontermination)
- Proposed solutions
  - symbol transition graph
  - generalized TCAP-unifiability
  - combination

Symbol transition graph (in NaTT & TTT2)

$$\mathcal{R} = \begin{cases} f(\ldots) \to g(\ldots) \\ g(\ldots) \to c(\ldots) \\ h(\ldots) \to x \end{cases}$$



- $c(\ldots) \to_\mathcal{R}^* t \quad \Rightarrow \quad t$ must be $c(\ldots)$
- $g(\ldots) \to_\mathcal{R}^* t \quad \Rightarrow \quad t$ must be $g(\ldots)$ or $c(\ldots)$
- $f(\ldots) \to_\mathcal{R}^* t \quad \Rightarrow \quad t$ must be $f(\ldots)$ or $g(\ldots)$ or $c(\ldots)$
- $h(\ldots) \to_\mathcal{R}^* t \quad \Rightarrow$ don't know

Theorem (to be formalized):

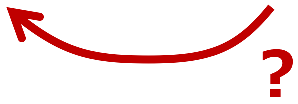
Define graph $G_\mathcal{R} = \langle \mathcal{F}, \sqsupset \rangle$ s.t. $f \sqsupset g$ whenever $f(\ldots) \to g(\ldots) \in \mathcal{R}$ or $f(\ldots) \to x \in \mathcal{R}$.
Then $f \sqsupset^* g$ is a complete estimation of $f(\ldots) \hookrightarrow_\mathcal{R} g(\ldots)$

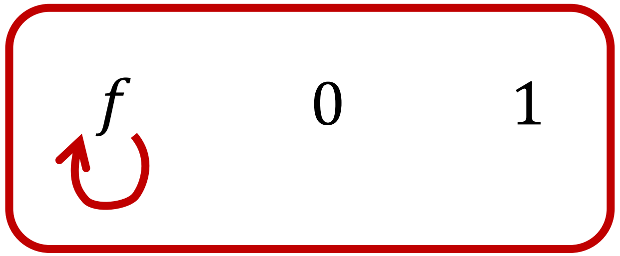
Symbol transition graph' (only in TTT2)

- Example:

$\mathcal{R} = \{\, f(0,1,x) \to f(x,x,x) \,\}$

$\mathcal{P} = \{\, f^{\#}(0,1,x) \to f^{\#}(x,x,x) \,\}$





$\boldsymbol{G_{\mathcal{R}}}$

reduced to "$\exists x.\ x \hookrightarrow_{\mathcal{R}} 0 \ \wedge\ x \hookrightarrow_{\mathcal{R}} 1 \ \wedge\ x \hookrightarrow_{\mathcal{R}} x'$"

... UNSAT, since 0 and 1 have no common ancestor in $\boldsymbol{G_{\mathcal{R}}}$

TODO: efficient algorithm for common ancestors (in graph)

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TCAP-unifiability [Giesl+'05]

- complete estimation of reachability:

$$s \hookrightarrow_{\mathcal{R}} t \Longrightarrow \mathrm{tcap}_{\mathcal{R}}(s) \sim_{\mathrm{unif}} t$$

Implementation in NaTT: $\mathrm{tcap}(s) \sim_{\mathrm{unif}} t \;\; \Rightarrow$

- $s \in \mathcal{V}$ or
- $t \in \mathcal{V}$ or
- $s = f(s_1, \dots, s_n)$ and
  - $t = f(t_1, \dots, t_n)$ and $\forall i.\, \mathrm{tcap}(s_i) \sim_{\mathrm{unif}} t_i$, or
  - $\exists f(l_1, \dots, l_n) \to r \in \mathcal{R}.\ \forall i.\, \mathrm{tcap}(s_i) \sim_{\mathrm{unif}} l_i$

TCAP-unifiability [Giesl+'05]

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  - $\exists f(l_1, \dots, l_n) \to r \in \mathcal{R}.\ \forall i.\, \mathrm{tcap}(s_i) \sim_{\mathrm{unif}} l_i$

TCAP-unifiability reformulated

- complete estimation of reachability:

$$s \hookrightarrow_{\mathcal{R}} t \Longrightarrow s \hookrightarrow_{\mathcal{R},1} t$$

Definition: $s \hookrightarrow_{\mathcal{R},1} t$ iff
- $s \in \mathcal{V}$ or
- $t \in \mathcal{V}$ or
- $s = f(s_1, \dots, s_n)$ and
  - $t = f(t_1, \dots, t_n)$ and $\forall i. s_i \hookrightarrow_{\mathcal{R},1} t_i$, or
  - $\exists f(l_1, \dots, l_n) \to r \in \mathcal{R}. \forall i. s_i \hookrightarrow_{\mathcal{R},1} l_i$

if $s \hookrightarrow_{\mathcal{R}} l \to r \in \mathcal{R}$ then give up

valid only if $r \hookrightarrow_{\mathcal{R}} t$

k-step look-ahead (only in NaTT)

- complete estimation of reachability:

$$s \hookrightarrow_{\mathcal{R}} t \Longrightarrow s \hookrightarrow_{\mathcal{R},k} t$$

Definition: $s \hookrightarrow_{\mathcal{R},k} t$ iff
- $s \in \mathcal{V}$ or
- $t \in \mathcal{V}$ or
- $s = f(s_1, \dots, s_n)$ and
  - $t = f(t_1, \dots, t_n)$ and $\forall i. s_i \hookrightarrow_{\mathcal{R},k} t_i$, or
  - $\exists f(l_1, \dots, l_n) \to r \in \mathcal{R}. \forall i. s_i \hookrightarrow_{\mathcal{R},k} l_i$ and $r \hookrightarrow_{\mathcal{R},k-1} t$

if $s \hookrightarrow_{\mathcal{R}} l \to r \in \mathcal{R}$, check $r \hookrightarrow_{\mathcal{R}} t$

- Experiments: ($k = 8$, empirically chosen)

  +10 YESs (all known, from MNZ_10)

Estimating reachability

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Combination (straightforward)

Definition: $s \hookrightarrow_{\mathcal{R},k} t$ iff

- $s \in \mathcal{V}$ or
- $t \in \mathcal{V}$ or
- $s = f(s_1, \dots, s_n)$ and
  - $t = f(t_1, \dots, t_n)$ and $\forall i.\, s_i \hookrightarrow_{\mathcal{R},k} t_i$, or
  - if $k = 0$ then use $\boldsymbol{G_{\mathcal{R}}}$
  - else $\exists f(l_1, \dots, l_n) \to r \in \mathcal{R}.\ \forall i.\, s_i \hookrightarrow_{\mathcal{R},k} l_i$ and $r \hookrightarrow_{\mathcal{R},k-1} t$

Conclusion

- (Almost) exact usable rules via reachability
- New reachability estimation
  - symbol transition graph
  - k-step look-ahead (generalizing TCAP-unifiability)
- TODO:
  - missing formalizations/implementations/evaluations
  - use substitution
  - combine with CTRS techniques [Sternagel & Sternagel '16]?